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Antoine Billot, Jean-Christophe Vergnaud, Bernard Walliser. Multiplayer belief revision. 2008. halshs-00344461

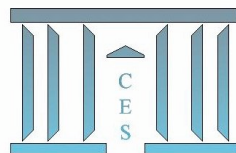
HAL Id: halshs-00344461

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Submitted on 4 Dec 2008

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Multiplayer belief revision

Antoine BILLOT, Jean-Christophe VERGNAUD, Bernard WALLISER

2008.67



CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

MULTIPLAYER BELIEF REVISION

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August 12, 2008

Abstract

One main topic of the epistemic program of game theory deals with the value of information. To study this question in a broad context, one needs to adapt some of the tools used in multiplayer epistemic logic. A hierarchical belief structure is introduced both in a syntactical and a semantical framework. In the same framework, a generalized notion of message is characterized by its content and its status. For a given message, a multiplayer belief revision rule that transforms any initial belief structure into a final belief one is designed. A representation theorem relates syntactical axioms to the belief revision rule.

Journal of Economic Literature Classification Number: C72, D82

1 Introduction

The information value was first introduced in decision theory by Blackwell (1951), then in game theory by Hirshleifer (1971). Consider a player who is uncertain about the states of nature, who receives a message which is correlated to the true state, who revises his beliefs according to the message and who chooses an (equilibrium) action according to his beliefs. The differential (average) utility he gets after receiving the message and before receiving the message is precisely called its ‘information value’.

The information value is always positive in decision theory under uncertainty, at least when the decision-maker maximizes his expected utility and receives a partitional message. Under the two last conditions, it may be negative in game theory for any player as shown in many examples (Hirschleifer, 1971; Kamien-Taumann-Zamir, 1990). However, in three different game cases, the information value is positive for the receiver of : (i) a secret message (Neyman, 1991); (ii) a private message in a zero-sum game (Gossner, 2001); (iii) a public message in a pure coordination game.

In this literature, the models are highly restricted. Firstly, while initial and final beliefs are exogeneously and explicitly given, the message linking both remains implicit and informal. Secondly, when one compares belief structures thanks to an accuracy order, this order is specifically designed for dealing with each type of message. Thirdly, the framework that is used is just semantical, without syntactical counterpart, and it is moreover limited to the partitional case.

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In order to overpass these limitations, the problem has to be more precisely analyzed in epistemic logics. Firstly, the message must be isolated and formally defined in terms of content (material, epistemic) and status (public, private, secret). Secondly, an accuracy order expressing that a belief structure is more informative than another must be set up by means of general conditions. Thirdly, the belief revision rule has to be given in syntax and in semantics and should also apply to non-partitional cases.

The present paper focuses on the logical foundations of the belief revision rule for hierarchical belief structures. All assumptions and results are given in syntax (using belief operators on propositions) and in semantics (using general accessibility relations on possible worlds). The belief revision rule is designed in order to infer a final belief from any initial belief and any message. Note that two subsequent papers will further introduce accuracy orders and investigate the question of the value of information.

In this paper, Section 2 presents a motivating example based on *Madame Bovary* story and followed all along. In Section 3, the syntactical and semantical definitions of a hierarchical belief structure are recalled. In Section 4, a message is defined within the same formal belief structure but interpreted in a different way. In syntax, the content of a message is some proposition of the belief language that is conveyed to each player and its status is described through propositions in a learning language which expresses who learns the content of the message and its diffusion. Then some specific types of message are stated in syntax and described formally in semantics. In Section 5, axioms are provided for multiplayer belief revision. The Belief Inference axiom states that a proposition is finally believed by a player if and only if this player gets some message and initially believes that the message involves the proposition. A symmetric Message Inference axiom is also stated, but the two axioms have to be slightly generalized in order to get the representation theorem. Actually, the corresponding semantical belief revision rule is exhibited as a cross product of the initial belief and the message leading to the final belief. In Section 6, the related works are examined.

2 Motivating Example

The literature is full of psychological situations in which characters have crossed beliefs one about another. All along the novels, messages are sent and received which lead to revise beliefs. For instance, in the famous French novel *Madame Bovary* (1856), Gustave Flaubert sets up his scenario around Emma's adultery. Since an adultery is not a public event in the French society of the XIXth century, crossed beliefs are quite natural, especially those of Emma and her husband Charles. Let us imagine the following Flaubert style example.

Just after Emma's first love affair with Rodolphe, the initial beliefs are the following. At the first level, Emma knows that she is unfaithful and her husband does not know but he has some doubts. At the second level, Emma believes either that Charles knows she is unfaithful or that Charles does not know and Charles obviously knows that his wife knows the truth, whatever it is. At the third level, Emma thinks that Charles thinks that she knows the truth and Charles thinks that Emma thinks that he believes she is faithful.

Suppose now that a message occurs... Charles learns the truth from Félicité, the homemaker. Suppose moreover that Emma, being hidden behind the door, hears Félicité at the very moment she speaks to Charles. Note that the message received by Charles is a first level one since it deals with the material situation, namely the fact that Emma is adultery. Moreover Charles

thinks that the message he received is secret because he meets Félicité alone. Of course Emma knows its content which then becomes shared. In return, the message received by Emma is a second level one since it deals with Charles' beliefs. Moreover, it is secret since Charles is not aware that his wife is behind the door.

Such a story does not lead to an immediate belief revision. Actually, the intuition works with difficulty (try yourself!) even though it follows the natural language and then is syntactical. Furthermore, there exists no revision method in the literature that is able to deal with such a situation characterized by the following features: beliefs and messages are syntactical; messages induce a simultaneous revision of all players' beliefs; the content of messages is material or epistemic; the status of messages is of any kind, especially secret; erroneous beliefs must be taken into account.

For clarity, the Bovary example will be formally treated all along the paper. The final belief will first be computed in a semantical framework then translated in syntax. The results are the following. At the two first levels, Emma and Charles know and know that the other knows the truth. At the third level, Emma knows that Charles knows that she knows the truth. But Charles still believes that Emma believes that he believes she is faithful. Finally, at the fourth level, Emma knows that Charles knows that Emma believes either that Charles knows the truth or that Charles doubts while Charles believes that Emma believes that he believes that she believes she is faithful.

3 Initial Belief

3.1 Syntax

Let I be a finite set of players i . A *multiplayer language* is defined according to three components:

- (i) *primitive propositions* typically labeled p, q, \dots that describe the common physical environment faced by the players and form a finite nonempty set \mathcal{P} ,
- (ii) *propositional connectors*, i.e. negation \neg , conjunction \wedge , material implication \rightarrow , disjunction \vee ,
- (iii) *epistemic operators*, denoted $(\mathbf{B}_i)_{i \in I}$, that describe players beliefs about physical environment and about other players beliefs.

The *multiplayer language* \mathcal{L} is a set of well-formed formulas typically labeled φ or ψ, \dots which is induced by three requirements:

$$\begin{cases} \text{if } p \in \mathcal{P}, \text{ then } p \in \mathcal{L}, \\ \text{if } \varphi, \psi \in \mathcal{L}, \text{ then } \neg\varphi, \neg\psi, \varphi \wedge \psi, \varphi \rightarrow \psi, \varphi \vee \psi \in \mathcal{L}, \\ \text{if } \varphi \in \mathcal{L}, \text{ then } \forall i \in I, \mathbf{B}_i\varphi \in \mathcal{L}. \end{cases} \quad (1)$$

Especially, $\mathcal{L}_0 \subset \mathcal{L}$ is the subset of all propositions without belief operator.

A SYntactic Structure (SYS) combines two classes of propositions: the tautological ones and the contingent ones.

First, consider the subset $\mathcal{T} \subset \mathcal{L}$ of propositions that are necessarily true. They are obtained by three axioms and two inference rules.¹

A1: All tautologies of propositional calculus.

¹The propositions $\varphi \in \mathcal{T}$ are provable and sometimes denoted $\vdash \varphi$ in the literature.

A2 (Logical Omniscience): $\mathbf{B}_i\varphi \wedge \mathbf{B}_i(\varphi \rightarrow \psi) \rightarrow \mathbf{B}_i\psi$.

A3 (Weak Consistency): $\neg\mathbf{B}_i\perp$.

R1 (Modus Ponens): From φ and $\varphi \rightarrow \psi$, infer ψ .

R2 (Necessitation Rule): From φ , infer $\mathbf{B}_i\varphi$.

Second, for a subset \mathcal{C} of \mathcal{L} , let us define two constraints:

C1 (No Contradiction): If $\varphi \in \mathcal{C}$, then $\neg\varphi \notin \mathcal{C}$.

C2 (Completeness): If $\varphi \notin \mathcal{C}$, then $\neg\varphi \in \mathcal{C}$.

The universe is described through a subset of propositions.

Definition 1 : A SYntactic Structure (SYS) is a subset \mathcal{K} of \mathcal{L} which satisfies **R1**, **C1** and **C2**, with $\mathcal{T} \subset \mathcal{K}$.

Note that we do not assume the usual axioms for epistemic operators, that is Veridicity (or Truth), Positive and Negative Introspection.

Bovary Example: In order to describe the Bovary belief-language, we need two operators, \mathbf{B}_E (Emma believes...) and \mathbf{B}_C (Charles believes...). The only primitive proposition is denoted p (Emma is unfaithful). The SYS \mathcal{K}_{Bovary} contains the following contingent propositions at the three first levels:

physical environment	
0-level	p
Emma's beliefs	
1-level	$\mathbf{B}_E p$
2-level	$\mathbf{B}_E \neg \mathbf{B}_C \neg p, \neg \mathbf{B}_E \mathbf{B}_C p, \neg \mathbf{B}_E \neg \mathbf{B}_C p$
3-level	$\mathbf{B}_E (\mathbf{B}_C \mathbf{B}_E p \vee ((\mathbf{B}_C (\mathbf{B}_E p \vee \mathbf{B}_E \neg p) \wedge \neg \mathbf{B}_C \mathbf{B}_E p \wedge \neg \mathbf{B}_C \mathbf{B}_E \neg p)))$
Charles' beliefs	
1-level	$\neg \mathbf{B}_C p, \neg \mathbf{B}_C \neg p$
2-level	$\neg \mathbf{B}_C \mathbf{B}_E p, \neg \mathbf{B}_C \mathbf{B}_E \neg p, \mathbf{B}_C ((p \rightarrow \mathbf{B}_E p) \wedge (\neg p \rightarrow \mathbf{B}_E \neg p))$
3-level	$\mathbf{B}_C \mathbf{B}_E \mathbf{B}_C \neg p$

3.2 Semantics

To represent the actual universe in semantics, we consider the Kripke model composed of possible worlds and accessibility relations between them. More precisely, we define a SEMantic Structure (SES)² by a list of its components plus a condition of Connectedness:

²In epistemic logics, a SES corresponds to a *connected pointed* belief model.

Definition 2 : A SEmanic Structure (SES) for a set I of players is defined as a 5-uple $\mathcal{H} = (W, (H_i)_{i \in I}, S, H_0, \mathbf{w})$ where:

- (i) W is a set of mutually exclusive possible worlds denoted w ,
- (ii) H_i is an accessibility relation from W toward $2^W \setminus \emptyset$,
- (iii) S is a set of mutually exclusive states of nature s ,
- (iv) H_0 is the accessibility relation of nature defined from W toward S ,
- (v) \mathbf{w} is the actual world

such that:

(Connectedness) In the actual world \mathbf{w} , for each world w , there exists a finite sequence of players i_1, \dots, i_n and a finite sequence of worlds w_0, \dots, w_n such that $w_0 = \mathbf{w}$, $w_n = w$ and for all $n \geq k \geq 1$, $w_k \in H_{i_k}(w_{k-1})$.

A (possible) world is a full description of the physical environment and of the players beliefs about it. The physical part called nature is represented by a state of nature. The psychical part representing the beliefs of the players about the physical world are embodied in the accessibility relations. They gather the worlds which are considered *undiscernable* by a given player in a given world. Connectedness means that the set of possible worlds does not include inaccessible worlds from the actual one.

Bovary Example: In order to describe the Bovary semantical structure, we just introduce two states of nature, p (Emma is unfaithful) and $\neg p$ (Emma is faithful). The SES \mathcal{H}_{Bovary} corresponds to the following graph.³ (Figure1(structureinitiale).pdf)

3.3 Transcription principles

The transcription principles link syntax and semantics. Consider a SES \mathcal{H} and define an *interpretation function* π as a mapping which associates to each state $s \in S$ a truth assignment of the primitive propositions, i.e. $\pi(s) : p \in \mathcal{P} \rightarrow \{true, false\}$.⁴ The truth value of a formula φ in some particular world w is written: $(\mathcal{H}, w) \models \varphi$, which means that φ is true in world w within the SES \mathcal{H} . Let $|\varphi|$ be the *field* of the formula φ , i.e. the set of worlds where φ is true. The following valuation rules express the truth assignment for any formula:

$$\left\{ \begin{array}{l} (\mathcal{H}, w) \models p \text{ iff } \pi \circ H_0(w)(p) = \text{true}, \\ (\mathcal{H}, w) \models \neg \varphi \text{ iff } (\mathcal{H}, w) \not\models \varphi, \\ (\mathcal{H}, w) \models \varphi \wedge \psi \text{ iff } (\mathcal{H}, w) \models \varphi \text{ and } (\mathcal{H}, w) \models \psi, \\ \left\{ \begin{array}{l} (\mathcal{H}, w) \models \mathbf{B}_i \varphi \text{ iff } (\mathcal{H}, w') \models \varphi \text{ for all } w' \text{ such that } w' \in H_i(w), \\ \text{i.e. } (\mathcal{H}, w) \models \mathbf{B}_i \varphi \text{ iff } H_i(w) \subseteq |\varphi|. \end{array} \right. \end{array} \right. \quad (2)$$

A SYS \mathcal{K} is associated with a SES \mathcal{H} by the following bridge principle:

$$\varphi \in \mathcal{K} \text{ iff } (\mathcal{H}, \mathbf{w}) \models \varphi. \quad (3)$$

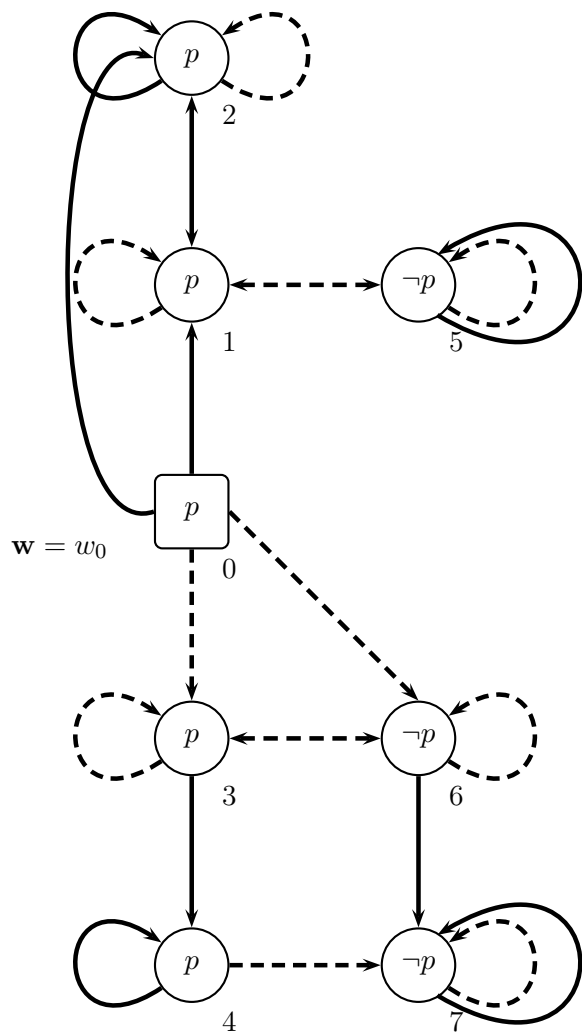
³The actual world $\mathbf{w} = w_0$ is represented by a square and the other worlds (w_1, \dots, w_7) by a circle. All of them are numbered as indexes. Besides, the state of each world appears inside. Emma's accessibility relation is represented by a full line while Charles' is a dotted line.

⁴The usual valuation function v is a composition of the accessibility relation of the nature H_0 with the interpretation function π : it associates, for each world, a truth assignment of the primitive propositions: $v(w) = \pi(H_0(w))$.

The SYS \mathcal{K} corresponds to all formulas which are true in the actual world \mathbf{w} . The Kripke representation theorem asserts that it is always possible to associate a set of SESs \mathcal{H} – all of them being bisimilar⁵ – to a SYS \mathcal{K} (van Benthem, 2000).

Bovary Example: For instance, let us check if $(\mathcal{H}_{Bovary}, \mathbf{w}) \models B_C B_E B_C \neg p$. First, note that, on Figure 1, $|\neg p| = \{w_5, w_6, w_7\}$. Then, $|B_C \neg p| = \{w_1, w_4, w_5, w_7\}$, $|B_E B_C \neg p| = \{w_3, w_4, w_6\}$. Finally, $|B_C B_E B_C \neg p| = \{w_0 = \mathbf{w}, w_3, w_4, w_6\}$.

⁵Two SESs are said to be *bisimilar* when they reflect the same syntactical propositions. Formal conditions for bisimilarity will be given in a subsequent paper.



—→ Emma
 - - -> Charles

4 Message

4.1 Syntax

Informally, a message is defined by two components. Its *content*, which describes the information received by each player (eventually different from one player to another), and its *status*, which describes what the players know about the diffusion of the information among players. Formally, from the point of view of the modeler, a message can be represented by a SYS in an auxiliary language (or a message language) denoted $\bar{\mathcal{L}}$. It is formed of three components:

- (i) the set of primitive propositions, $\mathcal{M} = \{m_0, \dots, m_l, \dots, m_L\}$, which is such that:
 - (i-a) for all l , $m_l \in \mathcal{L}$,
 - (i-b) for all $l \neq l'$, $m_l \wedge m_{l'} = \perp$ and
 - (i-c) $m_0 \vee m_1 \vee \dots \vee m_L = \top$,⁶
- (ii) the usual *propositional connectors*,
- (iii) the *epistemic operators* denoted $(\bar{\mathbf{B}}_i)_{i \in I}$ that describe players *learning* about the message and about what the other players learn.

Definition 3 : A message structure is a SYS $\bar{\mathcal{K}}$ defined on the language $\bar{\mathcal{L}}$.

More precisely, the content of a message received by an agent is a formula m in \mathcal{L} , however restricted to $\bar{\mathcal{L}}_0$. It is obtained as a combination of primitive messages m_l which jointly express the maximal degree of resolution of the message. The status of the message is given by all formulas containing at least one epistemic operator, contained in $\bar{\mathcal{L}} \setminus \bar{\mathcal{L}}_0$. For instance, the formula $\bar{\mathbf{B}}_i \bar{\mathbf{B}}_j m$ expresses that player i learns that player j learns the content m .

Bovary Example: The syntactical structure of the message $\bar{\mathcal{K}}_{Bovary}$ carried by Félicité is based on the following three first levels. The message m_0 stands for p and m_1 for $\neg p$:

physical environment	
0-level	$m_0 = p$

Emma's beliefs	
1-level	$\bar{\mathbf{B}}_E m_0$
2-level	$\bar{\mathbf{B}}_E \bar{\mathbf{B}}_C m_0$
3-level	$\bar{\mathbf{B}}_E \bar{\mathbf{B}}_C (\neg \bar{\mathbf{B}}_E m_0 \wedge \neg \bar{\mathbf{B}}_E m_1)$

Charles' beliefs	
1-level	$\bar{\mathbf{B}}_C m_0$
2-level	$\bar{\mathbf{B}}_C (\neg \bar{\mathbf{B}}_E m_0 \wedge \neg \bar{\mathbf{B}}_E m_1)$
3-level	$\bar{\mathbf{B}}_C \bar{\mathbf{B}}_E (\neg \bar{\mathbf{B}}_C m_0 \wedge \neg \bar{\mathbf{B}}_C m_1)$

⁶This condition on the set of primitive propositions for the auxiliary language $\bar{\mathcal{L}}$ seems more demanding than the corresponding condition for the set of primitive propositions for the language \mathcal{L} . It is not true. Indeed, the set of primitive propositions is arbitrary and it is always possible to use a set which satisfies conditions (i-a) and (i-b). We introduce these conditions because the description of the content of a message is definitely more intuitive.

4.2 Semantics

The SYS $\bar{\mathcal{K}}$ has a semantical counterpart given by a SES $\bar{\mathcal{H}} = (\bar{W}, (\bar{H}_i)_{i \in I}, \bar{S}, \bar{H}_0, \bar{\mathbf{w}})$. The set of states of nature $\bar{S} = \{\bar{s}_0, \dots, \bar{s}_l, \dots, \bar{s}_L\}$ is such that the interpretation function $\bar{\pi}$ is defined as $\bar{\pi}(\bar{s}_l)[m_l] = \text{true}$, while $\bar{\pi}(\bar{s}_l)[m_{l'}] = \text{false}$, for $l \neq l'$. Hence, each state of nature corresponds to a primitive proposition in \mathcal{M} . For a syntactical content m in \mathcal{L} and $\bar{\mathcal{L}}_0$, denote $M = |m| \subset W$ its semantical content in the initial SES and $\bar{M} = |m| \subset \bar{W}$ its semantical content in the auxiliary SES $\bar{\mathcal{H}}$.

Bovary Example: The semantical structure of the message is given by the following graph:
(Figure2(message).pdf)

4.3 Types of Message

Four different standard types of message are usually considered, according to their different status. They are informally defined in syntax and formally defined in semantics.

- In a *public message*, it is commonly learned that each player receives the content m_0 :

Definition 4 : A public message is defined by the SES $\bar{\mathcal{H}}_{pub}$ where

$$\begin{cases} \bar{W} = \{\bar{\mathbf{w}}\}; \bar{\mathbf{w}} \in \bar{M}_0, \\ \forall i \in I, \bar{H}_i(\bar{\mathbf{w}}) = \{\bar{\mathbf{w}}\}. \end{cases}$$

- In a *private message*, one player receives a message of content m_0 while other players receive nothing, and this is commonly learned.

Definition 5 : A private message for i is defined by the SES $\bar{\mathcal{H}}_{pri}$ where

$$\begin{cases} \bar{W} = \{\bar{\mathbf{w}}, \bar{w}_1, \dots, \bar{w}_l, \dots, \bar{w}_L\}; \bar{\mathbf{w}} \in \bar{M}_0; \forall l \neq 0, \bar{w}_l \in \bar{M}_l, \\ \bar{H}_i(\bar{\mathbf{w}}) = \{\bar{\mathbf{w}}\}; \forall l \neq 0, \bar{H}_i(\bar{w}_l) = \{\bar{w}_l\} \text{ and,} \\ \forall j \neq i, \forall l \neq 0, \bar{H}_j(\bar{\mathbf{w}}) = \bar{H}_j(\bar{w}_l) = \{\bar{\mathbf{w}}, \bar{w}_1, \dots, \bar{w}_L\}. \end{cases}$$

- In a *secret message*, one player receives a message of content m_0 , the other players learn that nothing was learned and the first player learns it:

Definition 6 : A secret message for i is defined by the SES $\bar{\mathcal{H}}_{sec}$ where

$$\begin{cases} \bar{W} = \{\bar{\mathbf{w}}, \bar{w}_0, \bar{w}_1, \dots, \bar{w}_L\}; \bar{\mathbf{w}}, \bar{w}_0 \in \bar{M}_0; \forall l \neq 0, \bar{w}_l \in \bar{M}_l, \\ \bar{H}_i(\bar{\mathbf{w}}) = \{\bar{\mathbf{w}}\}; \forall l \neq 0, \bar{H}_i(\bar{w}_0) = \bar{H}_i(\bar{w}_l) = \{\bar{w}_0, \bar{w}_1, \dots, \bar{w}_L\} \text{ and,} \\ \forall j \neq i, \forall l \neq 0, \bar{H}_j(\bar{\mathbf{w}}) = \bar{H}_j(\bar{w}_0) = \bar{H}_j(\bar{w}_l) = \{\bar{w}_0, \bar{w}_1, \dots, \bar{w}_L\}. \end{cases}$$

- In a *null message*, it is commonly learned that nobody learns any content:

Definition 7 : A null message is defined by the SES $\bar{\mathcal{H}}_{nul}$ where

$$\begin{cases} \bar{W} = \{\bar{\mathbf{w}}, \bar{w}_1, \dots, \bar{w}_N\}; \bar{\mathbf{w}} \in \bar{M}_0; \forall l \neq 0, \bar{w}_l \in \bar{M}_l, \\ \forall i \in I, \forall l \neq 0, \bar{H}_i(\bar{\mathbf{w}}) = \bar{H}_i(\bar{w}_l) = \{\bar{\mathbf{w}}, \bar{w}_1, \dots, \bar{w}_L\}. \end{cases}$$

But all possible types of message are not described within these four basic types. Other types can be proposed:

- In a *quasi-secret message*, one player learns a message of content m_0 but other players ignore if he has learned it or not.

Definition 8 : A quasi-secret message for i is defined by the SES $\overline{\mathcal{H}}_{qsec}$ where

$$\left\{ \begin{array}{l} \overline{W} = \{\overline{w}, \overline{w}_1, \dots, \overline{w}_L, \overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\}; \overline{w}, \overline{w}'_0 \in \overline{M}_0; \forall l \neq 0, \overline{w}_l, \overline{w}'_l \in \overline{M}_l, \\ \overline{H}_i(\overline{w}) = \{\overline{w}\}; \forall l \neq 0, \overline{H}_i(\overline{w}_l) = \{\overline{w}_l\}; \forall l \neq 0, \overline{H}_i(\overline{w}'_0) = \overline{H}_i(\overline{w}'_l) = \{\overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\} \text{ and,} \\ \forall j \neq i, \forall l \neq 0, \overline{H}_j(\overline{w}) = \overline{H}_j(\overline{w}'_0) = \overline{H}_j(\overline{w}_l) = \overline{H}_j(\overline{w}'_l) = \{\overline{w}, \overline{w}_1, \dots, \overline{w}_L, \overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\}. \end{array} \right.$$

- In a *Bcc message*,⁷ two players i, j learn a message of content m_0 but i is the only player to learn who receives it and this is commonly learned.

Definition 9 : A Bcc message for i is defined by the SES $\overline{\mathcal{H}}_{Bcc}$ where

$$\left\{ \begin{array}{l} \overline{W} = \{\overline{w}, \overline{w}_1, \dots, \overline{w}_L, \overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\}; \overline{w}, \overline{w}'_0 \in \overline{M}_0; \forall l \neq 0, \overline{w}_l, \overline{w}'_l \in \overline{M}_l, \\ \overline{H}_i(\overline{w}) = \{\overline{w}\}; \forall l \neq 0, \overline{H}_i(\overline{w}_l) = \{\overline{w}_l\}; \forall l \neq 0, \overline{H}_i(\overline{w}'_0) = \overline{H}_i(\overline{w}'_l) = \{\overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\} \text{ and,} \\ \forall j \neq i, \overline{H}_j(\overline{w}) = \overline{H}_j(\overline{w}'_0) = \{\overline{w}, \overline{w}'_0\}; \forall l \neq 0, \overline{H}_j(\overline{w}_l) = \overline{H}_j(\overline{w}'_l) = \{\overline{w}_l, \overline{w}'_l\}. \end{array} \right.$$

- In a *private message believed public*, only one player learns a message of content m_0 but he learns wrongly that other players also learn it.

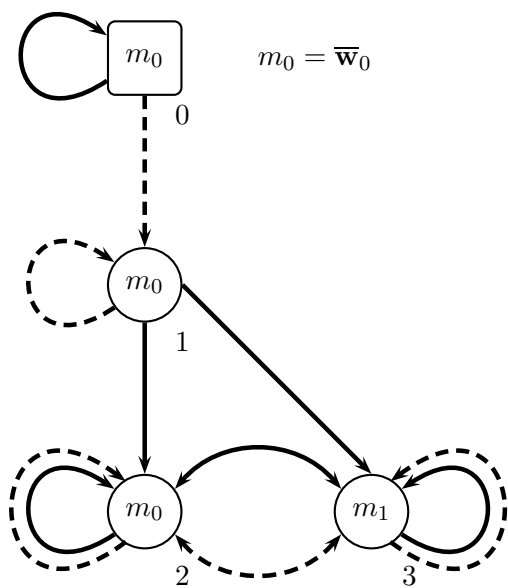
Definition 10 : A private message believed public for i is defined by the SES $\overline{\mathcal{H}}_{ppr}$ where

$$\left\{ \begin{array}{l} \overline{W} = \{\overline{w}, \overline{w}_1, \dots, \overline{w}_L, \overline{w}'_0, \overline{w}'_1, \dots, \overline{w}'_L\}; \overline{w}, \overline{w}'_0 \in \overline{M}_0; \forall l \neq 0, \overline{w}_l, \overline{w}'_l \in \overline{M}_l, \\ \overline{H}_i(\overline{w}) = \overline{H}_i(\overline{w}'_0) = \{\overline{w}'_0\}; \forall l \neq 0, \overline{H}_i(\overline{w}_l) = \overline{H}_i(\overline{w}'_l) = \{\overline{w}'_l\} \text{ and,} \\ \forall j \neq i, \forall l \neq 0, \overline{H}_j(\overline{w}) = \overline{H}_j(\overline{w}_l) = \{\overline{w}, \overline{w}_1, \dots, \overline{w}_L\}; \overline{H}_j(\overline{w}'_0) = \{\overline{w}'_0\}; \forall l \neq 0, \overline{H}_j(\overline{w}'_l) = \{\overline{w}'_l\}. \end{array} \right.$$

Bovary Example: From the point of view of Charles, the message concerning the physical environment (Emma is unfaithful) is secret while actually the content of the message is shared. From the point of view of Emma, the message concerning the message received by Charles is secret and actually, it is!⁸

⁷‘Bcc’ means ‘Blind Carbon Copy’ in electronic mail language.

⁸In the offspring paper which will define accuracy orders, these intuitions will be precisely characterized.



$$m_0 = \overline{\mathbf{w}}_0$$

—→ Emma
 - - -> Charles

5 Multiplayer Belief Revision

5.1 Syntactic Foundations

Denote \mathcal{L}^* the language in which the final belief is expressed. \mathcal{L}^* is defined according to:

- (i) the *primitive propositions* \mathcal{P} which are the same in \mathcal{L}^* than in \mathcal{L} ,
- (ii) the *epistemic operators* which are denoted \mathbf{B}_i^* .

Denote $\mathbb{L} = (\mathcal{L}, \bar{\mathcal{L}}, \mathcal{L}^*)$ the language which extends $\mathcal{L} \cup \bar{\mathcal{L}} \cup \mathcal{L}^*$:

$$\begin{cases} \text{if } \varphi, \psi \in \mathbb{L}, \text{ then } \neg\varphi, \neg\psi \text{ and } (\varphi \wedge \psi) \in \mathbb{L}, \\ \text{if } \varphi \in \mathbb{L}, \text{ then } \forall i \in I, \mathbf{B}_i^*\varphi \in \mathbb{L}. \end{cases} \quad (4)$$

This language allows simultaneously to deal with propositions belonging to the initial belief, to the message and to the final belief as well as with propositions of the final belief over initial belief or message.

We define a *Multioperator SYS* (denoted *MSYS*) by extension of a SYS. We consider first a subset $\mathbb{T} \subset \mathbb{L}$ of tautological propositions that are obtained by axioms **A1-A3** (for the three operators $\mathbf{B}_i, \bar{\mathbf{B}}_i, \mathbf{B}_i^*$) and inference rules **R1** and **R2** [from $\varphi \in \mathcal{L}$ (resp. $\bar{\mathcal{L}}$ (resp. \mathbb{L})), infer $\mathbf{B}_i\varphi$ (resp. $\bar{\mathbf{B}}_i\varphi$ (resp. $\mathbf{B}_i^*\varphi$))]. Then we consider a subset $\mathbb{C} \subset \mathbb{L}$ of contingent propositions, with $\mathbb{T} \subset \mathbb{K}$, that are true in the actual universe and which are submitted to Modus Ponens **R1**, No Contradiction **C1** and Completeness **C2** (cf. §3.1). The *MSYS* is a subset \mathbb{K} of \mathbb{L} which satisfies **R1**, **C1** and **C2** and contains $\mathbb{T} \cup \mathbb{C}$.

The interpretation of the *MSYS* \mathbb{K} is the following: $\mathbb{K} \cap \mathcal{L}$ is a SYS in \mathcal{L} interpreted as the initial belief \mathcal{K} , $\mathbb{K} \cap \bar{\mathcal{L}}$ is a SYS in $\bar{\mathcal{L}}$ interpreted as the message $\bar{\mathcal{K}}$ and $\mathbb{K} \cap \mathcal{L}^*$ is a SYS in \mathcal{L}^* which will be interpreted as the final belief \mathcal{K}^* .

In order to characterize the revision process, we impose two following axioms to the *MSYS* \mathbb{K} :

A4 (No Conflict): $\bar{\mathbf{B}}_i m \rightarrow \neg \mathbf{B}_i \neg m$.

No Conflict means that the message does not contradict the initial belief.

A5 (Belief and Message Inference): For any φ in $\mathbb{K} \cap \mathcal{L}$ and any ψ in $\mathbb{K} \cap \bar{\mathcal{L}}$,

$$\bigwedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i (m_l \rightarrow \varphi) \vee \bar{\mathbf{B}}_i (m_l \rightarrow \psi)) \longleftrightarrow \mathbf{B}_i^* (\varphi \vee \psi).$$

This intricate axiom has meaningful consequences. Consider that ψ is a tautology, hence,

$$\bigwedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l \vee (\mathbf{B}_i (m_l \rightarrow \varphi))) \longleftrightarrow \mathbf{B}_i^* (\varphi).$$

Consider moreover that the message the player i learns is the basic message m_s . It follows that $\bar{\mathbf{B}}_i m_s$ and, by No Conflict, $\neg \mathbf{B}_i \neg m_s$. For $l \neq s$, the terms in the conjunction are tautologies (since $\bar{\mathbf{B}}_i \neg m_l$ is true) and for $l = s$, the term reduces to $\mathbf{B}_i (m_s \rightarrow \varphi)$ (since $\mathbf{B}_i \neg m_s \vee \bar{\mathbf{B}}_i \neg m_s$ is a contradiction). It follows that:

$$\bar{\mathbf{B}}_i m_s \wedge \mathbf{B}_i (m_s \rightarrow \varphi) \rightarrow \mathbf{B}_i^* \varphi.$$

It is easy to prove that the same relation applies to any message m .

The same operation can be done by exchanging initial belief and message. Hence, the Belief and Message Inference axiom entails the two following consequences.

$$\bigvee_{m \in \bar{\mathcal{L}}_0} \bar{\mathbf{B}}_i m \wedge \mathbf{B}_i (m \longrightarrow \varphi) \rightarrow \mathbf{B}_i^* \varphi. \quad (\mathbf{A5a} \text{ (Belief Inference)})$$

Belief Inference describes precisely the belief revision process as some kind of Modus Ponens: A proposition is finally believed iff the player accepts some message and believes initially that this message entails the proposition.

$$\bigvee_{m \in \bar{\mathcal{L}}_0} (\mathbf{B}_i m \wedge \bar{\mathbf{B}}_i (m \longrightarrow \psi)) \rightarrow \mathbf{B}_i^* \psi. \quad (\mathbf{A5b} \text{ (Message Inference)})$$

Message Inference means that a proposition is finally believed iff the player who already knows some message learns that this message entails that proposition. Note that, when m is a tautology, this axiom moreover entails that any proposition initially believed and any proposition which is learnt is finally believed.

These axioms are considered common belief according to belief operators \mathbf{B}_i^* .

The following theorem ensures the unicity of the MSYS \mathbb{K} stemmed from the SYSs \mathcal{K} and $\bar{\mathcal{K}}$ and the axioms.

Theorem 1 : Consider two MSYSs $\mathbb{K} = (\mathcal{K}, \bar{\mathcal{K}}, \mathcal{K}^*)$ and $\mathbb{K}' = (\mathcal{K}', \bar{\mathcal{K}}', \mathcal{K}'^*)$ satisfying No Conflict **A4** and Belief and Message Inference **A5**. If $\mathcal{K} = \mathcal{K}'$ and $\bar{\mathcal{K}} = \bar{\mathcal{K}}'$ then $\mathcal{K}^* = \mathcal{K}'^*$.

Proof is in Appendix A.

Observe that there exists a unique final belief \mathcal{K}^* associated to \mathbb{K} .

Bovary Example: At this step, we just give an example of the way the conditions work. Lets prove that $\mathbf{B}_E^* \mathbf{B}_C^* \mathbf{B}_{Ep}$ holds. By Belief Inference **A5a**, $(\bar{\mathbf{B}}_C m_0 \wedge \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep})) \longrightarrow \mathbf{B}_C^* \mathbf{B}_{Ep}$. Since **A5a** is common belief, $\mathbf{B}_E^* ((\bar{\mathbf{B}}_C m_0 \wedge \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep})) \longrightarrow \mathbf{B}_C^* \mathbf{B}_{Ep})$. By Message Inference **A5b** with tautology as message, $\bar{\mathbf{B}}_E \bar{\mathbf{B}}_C m_0 \longrightarrow \mathbf{B}_E^* \bar{\mathbf{B}}_C m_0$ and since by hypothesis $\bar{\mathbf{B}}_E \bar{\mathbf{B}}_C m_0$ holds, then $\mathbf{B}_E^* \bar{\mathbf{B}}_C m_0$ is also true. By **A5a** with tautology as message, $\mathbf{B}_E \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep}) \longrightarrow \mathbf{B}_E^* \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep})$ and since by hypothesis $\mathbf{B}_E \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep})$ holds, then $\mathbf{B}_E^* \mathbf{B}_C (m_0 \longrightarrow \mathbf{B}_{Ep})$ is also true. Therefore by Modus Ponens **R1**, $\mathbf{B}_E^* \mathbf{B}_C^* \mathbf{B}_{Ep}$. Finally, once more by **A5a**, $\mathbf{B}_E^* \mathbf{B}_C^* (\mathbf{B}_{Ep} \longrightarrow \mathbf{B}_{Ep}^*)$, and by **R1**, we get the result.

5.2 Semantic Rule

Let us combine an initial SES

$$\mathcal{H} = (W, (H_i)_{i \in I}, S, H_0, \mathbf{w})$$

and an auxiliary SES

$$\bar{\mathcal{H}} = (\bar{W}, (\bar{H}_i)_{i \in I}, \bar{S}, \bar{H}_0, \bar{\mathbf{w}})$$

in order to get a final product structure denoted \mathcal{H}^* :

$$\mathcal{H}^* = \mathcal{H} * \bar{\mathcal{H}} = (W^*, (H_i^*)_{i \in I}, S^*, H_0^*, \mathbf{w}^*).$$

Definition 11 (Product Structure) : From an initial belief SES \mathcal{H} and a message SES $\overline{\mathcal{H}}$, the product structure \mathcal{H}^* is generated as follows:

Step 1: Define $\widetilde{\mathcal{H}}^* = (\widetilde{W}^*, (H_i^*)_{i \in I}, S^*, H_0^*, \mathbf{w}^*)$ such that

- (i) $\widetilde{W}^* = W \times \overline{W}$,
- (ii) $\forall (w, \overline{w}) \in \widetilde{W}^*, \forall i \in I, H_i^*(w, \overline{w}) = \bigcup_{l=0,1,\dots,L} ((H_i(w) \cap M_l) \times (\overline{H}_i(\overline{w}) \cap \overline{M}_l))$,
- (iii) $S^* = S$,
- (iv) $\forall (w, \overline{w}) \in \widetilde{W}^*, H_0^*(w, \overline{w}) = H_0(w)$,
- (v) $\mathbf{w}^* = (\mathbf{w}, \overline{\mathbf{w}})$.

Step 2: \mathcal{H}^* is the connected part of $\widetilde{\mathcal{H}}^*$.

In the product structure, each final world is obtained as a Cartesian product of an initial world and a message world. For any final world, its accessibility domain proceeds from the elimination of all initial and message worlds which contradict the same basic message. The set of states of nature remains. The state of nature associated to each initial world is kept in the corresponding final world. The final actual world is just a combination of the initial and message actual worlds.

One can observe that \mathcal{H}^* is a SES as soon as the accessibility relations $H_i^*(w, \overline{w})$ are non empty.

5.3 Representation Theorem

The following representation theorem can be proved:

Theorem 2 : Consider two SESs \mathcal{H} and $\overline{\mathcal{H}}$ and let the SYSs \mathcal{K} and $\overline{\mathcal{K}}$ be their syntactical counterpart.

(1) If $\mathcal{H}^* = \mathcal{H} * \overline{\mathcal{H}}$ is a SES, then there exists a MSYS $\mathbb{K} = (\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^*)$ which satisfies No Conflict **A4** and Belief and Message Inference **A5** and such that \mathcal{K}^* is the syntactical counterpart of \mathcal{H}^* .

(2) If there exists a MSYS $\mathbb{K} = (\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^*)$ which satisfies No Conflict **A4** and Belief and Message Inference **A5**, then $\mathcal{H}^* = \mathcal{H} * \overline{\mathcal{H}}$ is a SES and is a semantical counterpart of \mathcal{K}^* .

Proof is in Appendix B.

Bovary Example: We apply the product structure operation as follows. We consider all couples of worlds obtained by combining a world of the initial belief and a world of the message. We first define the accessibility relations stemming from the actual final world. We then do the same from the worlds which were accessible from the actual world. We proceed sequentially until no new world can be reached. The result is the final belief structure given in (Figure3(intermedstructurefinale).pdf). This last structure can be simplified as follows. The worlds $w_{1'}$ and $w_{2'}$ are equivalent to w_1 and w_2 (they have same accessibility domains). Hence, this graph is bisimilar to a simplified one⁹ where the equivalent worlds gather. This simplified graph is given in (Figure4(simplstructurefinale).pdf). We can deduce from the previous graph the syntactical belief composed of the following propositions at the first four levels.

⁹Bisimilarity will be define precisely in the offspring paper on accuracy orders.

physical environment

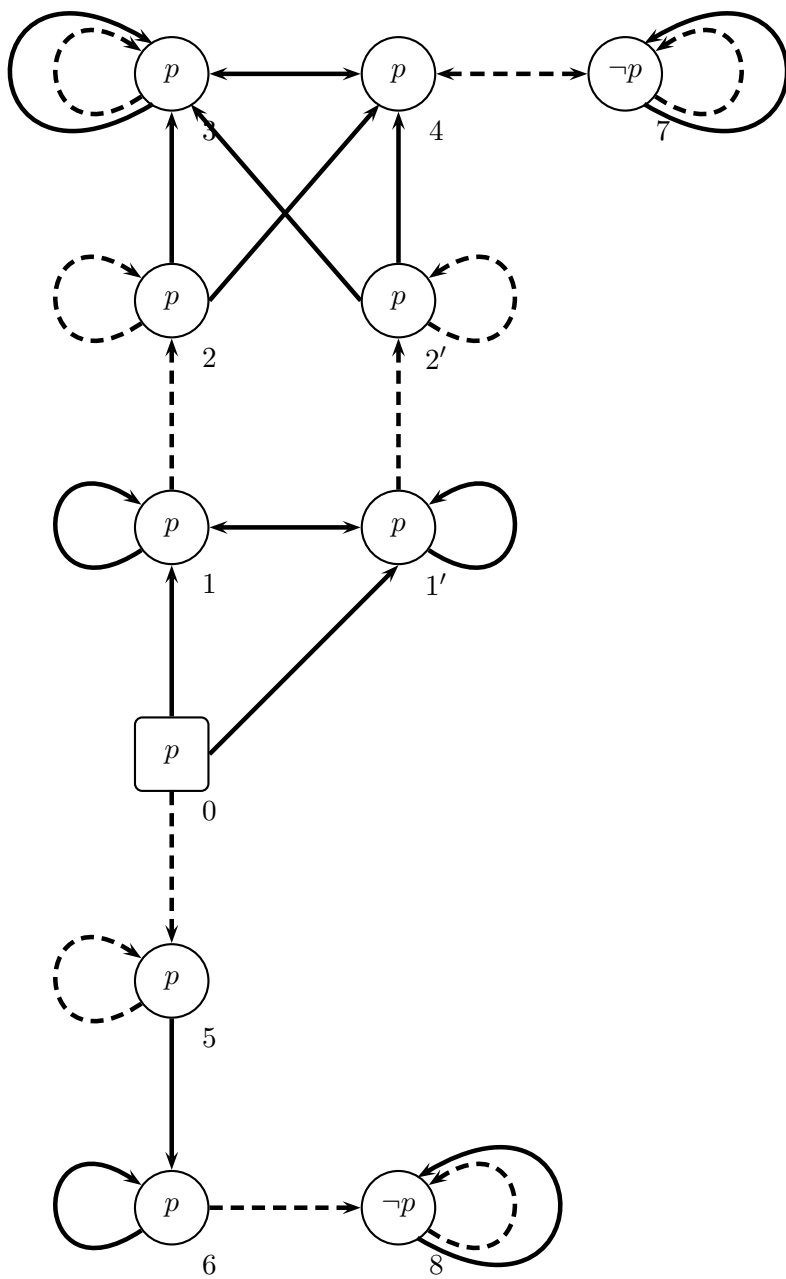
0-level	p
---------	-----

Emma's beliefs

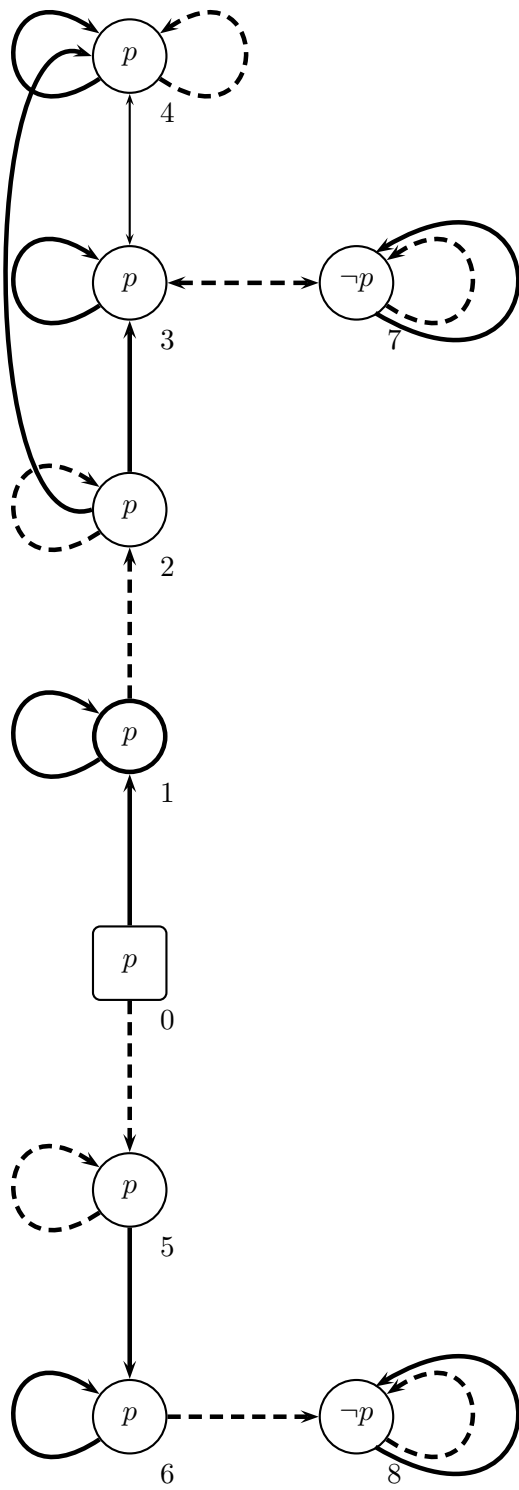
1-level	$\mathbf{B}_E^* p$
2-level	$\mathbf{B}_E^* \mathbf{B}_C^* p$
3-level	$\mathbf{B}_E^* \mathbf{B}_C^* \mathbf{B}_E^* p$
4-level	$\mathbf{B}_E^* \mathbf{B}_C^* (\mathbf{B}_E^* (\mathbf{B}_C^* p \vee (\neg \mathbf{B}_C^* p \wedge \mathbf{B}_C^* \neg p)))$

Charles' beliefs

1-level	$\mathbf{B}_C^* p$
2-level	$\mathbf{B}_C^* \mathbf{B}_E^* p$
3-level	$\mathbf{B}_C^* \mathbf{B}_E^* \mathbf{B}_C^* \neg p$
4-level	$\mathbf{B}_C^* \mathbf{B}_E^* \mathbf{B}_C^* \mathbf{B}_E^* \neg p$



—→ Emma
 - - -> Charles



—→ Emma
 - - -> Charles

6 Related works

6.1 Signals

In the signaling literature (Blackwell, 1951), messages are introduced thanks to signals. More precisely, consider a set of signals $Y = \prod_{i \in I} Y_i$. The set Y is finite and $Y = \{y^0, y^1, \dots, y^l, \dots, y^L\}$. A signal function $\sigma : W \rightarrow Y$ associates y to each world w where the i -component $\sigma_i(w)$ denotes the signal received by player i . The most usual types of signals are defined as follows. A public signal can be written as: $\forall i, j, \forall w, \sigma_i(w) = \sigma_j(w)$. A private signal for player i can be written as: $\forall j \neq i, \forall w, w', \sigma_j(w) = \sigma_j(w')$.

A revision rule is likely proposed. For each message y , it associates a final accessibility domain $H_i^y(w)$ to the initial accessibility domain $H_i(w)$ by the following rule:

$$H_i^y(w) = H_i(w) \cap \sigma_i^{-1}[\sigma_i(w)].$$

In order to compare this framework with ours, the following transposition is proposed. It associates a message SES to each signal Y as follows:

- (i) $M_l = \sigma^{-1}(y^l)$,
- (ii) $\bar{S} = \{\bar{s}_0, \bar{s}_1, \dots, \bar{s}_L\}$,
- (iii) $\bar{W} = \{\bar{w}_0, \bar{w}_1, \dots, \bar{w}_L\}$,
- (iv) $\forall \bar{w}_l, \bar{H}_0(\bar{w}_l) = \bar{s}_l$,
- (v) $\forall i, \forall \bar{w}_l, H_i(\bar{w}_l) = \{\bar{w}_{l'} : \sigma_i^{-1}(y_i^{l'}) = \sigma_i^{-1}(y_i^l)\}$.

The two types of messages (public, private) coincide with this transcription principle. Moreover, the belief revision rule is the same.

For a public message, $H_i^y(w) = H_i(w) \cap \sigma_i^{-1}[y]$. In our framework, since $\bar{W} = \{\bar{w}\}$, this can be rewritten as: $H_i^*(w, \bar{w}) = \{H_i(w) \cap M\} \times \{\bar{w}\}$. There obviously exists an isomorphism between $W \times \bar{W}$ and W . By using this isomorphism, $H_i^*(w, \bar{w}) =_{def} H_i^{**}(w) = H_i(w) \cap M$. Since M is the subset of worlds where the message is true, $H_i^{**}(w) = H_i^y(w)$.

For a private message for player i , $H_i^y(w) = H_i(w) \cap \sigma_i^{-1}[\sigma_i(w)]$ and $H_j^y(w) = H_j(w) \cap W = H_j(w)$. In our framework, $H_i^*(w, \bar{w}) = \{H_i(w) \cap M\} \times \{\bar{w}\}$ and $H_j^*(w, \bar{w}) = \{H_j(w) \cap M\} \times \bar{W}$. By using connectedness, the only worlds to remain are the worlds (\mathbf{w}, \bar{w}_k) . There obviously exists an isomorphism between $\{\mathbf{w}\} \times \bar{W}$ and W . By using this isomorphism, $H_i^*(w, \bar{w}) =_{def} H_i^{**}(w) = H_i(w) \cap M$ and $H_j^*(w, \bar{w}) =_{def} H_j^{**}(w) = H_j(w)$. Hence, $H_i^{**}(w) = H_i^y(w)$ and $H_j^{**}(w) = H_j^y(w)$.

However, some of the messages that can be defined in our framework have no counterpart in terms of signal. This is obviously the case of messages involving errors like a secret one. It is true as well for messages involving no errors like a quasi-secret or a Bcc one. This shows that our framework is more general than the usual one.

6.2 Epistemic belief revision rules

The main attempt to design a belief revision rule in epistemic logics is due to Alchourron-Gärdenfors-Makinson (1985) and is called the AGM system. It concerns only a single agent, but is adapted to the case where the message contradicts the initial belief. In syntax, AGM propose eight axioms concerning the transformation of the initial set of propositions into a revised one. In semantics, the final worlds are obtained by considering the worlds in the message which are the nearest from those of the initial belief. In syntax, it means that, whence a message is

added to the initial propositions, the less entrenched propositions (according to some epistemic entrenchment order) are deleted until restoring the consistency of the system. Note that the epistemic operators are not specifically introduced.

Bonanno (2005) still considers one player but he is endowed with three syntactical operators: \mathbf{B} (initial belief), $\overline{\mathbf{B}}$ (message) and \mathbf{B}^* (final belief). Besides, he restricts semantics to a unique world space which may be simultaneously interpreted as $W \times \overline{W}$ and W^* , the accessibility relation being respectively H , \overline{H} and H^* . His *belief axioms* are similar to ours.

Board (2004) considers several players i with an infinity of syntactical operators \mathbf{B}_i^m where m is any message that may contradict the initial belief. He also restricts semantics to a unique world space and sets up an entrenchment order over that space. Note that his *informational axiom* is similar to our two inference axioms, **A5a** and **A5b**. Besides, his *success axiom*, which expresses that $\overline{\mathbf{B}}_i m \rightarrow \mathbf{B}_i^* m$, can be deduced from Belief Inference **A5a** and Non-Conflict **A4**.

Baltag and *alii* (1998) whose approach is very close to ours, consider several players i who may receive any message of any kind. The proposed syntax expresses the player's belief after receiving a message in a dynamic logics framework. Their *action knowledge axiom* is similar to our inference axioms, **A5a** and **A5b**, but it needs to be written as a precondition on the message. Since we consider the subclass of messages which can be viewed as an auxiliary belief structure, the Baltag's precondition becomes explicit. The semantic belief revision rule becomes directly operational.

Aucher (2008) proposes a multi-agent belief revision theory, but it is restricted to the case of a secret message (which he calls private). In that case, the others' beliefs being unchanged, he can apply the AGM framework, even when the message contradicts the initial belief.

7 Acknowledgements

We thank A. Baltag, G. Bonanno, I. Gilboa, J. Lang, J.M. Tallon, J. van Benthem and S. Zamir for helpful remarks and comments (especially during conferences and seminars). We also thank L. Ménager for her help in designing the figures.

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9 Appendix A

Theorem 1 : Consider two MSYSs $\mathbb{K} = (\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^*)$ and $\mathbb{K}' = (\mathcal{K}', \overline{\mathcal{K}'}, \mathcal{K}'^*)$ satisfying No Conflict **A4** and Belief and Message Inference **A5**. If $\mathcal{K} = \mathcal{K}'$ and $\overline{\mathcal{K}} = \overline{\mathcal{K}'}$ then $\mathcal{K}^* = \mathcal{K}'^*$.

Proof. : Define the *depth* of a formula φ in \mathbb{L} , denoted $d(\varphi)$, as the number of modal operators \mathbf{B}_i^* (and not \mathbf{B}_i) that are hierarchically used. It is defined recursively by $d(\varphi) = 0$ if $\varphi \in \mathcal{L} \cup \overline{\mathcal{L}}$, $d(\neg\varphi) = d(\varphi)$, $d(\varphi \wedge \psi) = \max(d(\varphi), d(\psi))$ and $d(\mathbf{B}_i^* \varphi) = d(\varphi) + 1$. We are going to prove that $\mathbb{K} = \mathbb{K}'$. We proceed by induction on the depth of proposition in \mathbb{L} . Let \mathbb{L}_α be the set of propositions of \mathbb{L} whose depth is inferior or equal to α . Consider first propositions of depth 0. Note that \mathbb{L}_0 is the closure of $\mathcal{L} \cup \overline{\mathcal{L}}$ with respect to the propositionnal connectors. Remark that $\mathbb{K} \cap (\mathcal{L} \cup \overline{\mathcal{L}}) = \mathcal{K} \cup \overline{\mathcal{K}} = \mathcal{K}' \cup \overline{\mathcal{K}'} = \mathbb{K}' \cap (\mathcal{L} \cup \overline{\mathcal{L}})$. Since \mathbb{K} and \mathbb{K}' satisfy **R1**, $\mathbb{K} \cap \mathbb{L}_0 = \mathbb{K}' \cap \mathbb{L}_0$. Consider now \mathbb{L}_1 and propositions of the form $\mathbf{B}_i^*(\varphi \vee \psi)$ where $\varphi \in \mathcal{L}$ and $\psi \in \overline{\mathcal{L}}$. By **A5**, $\mathbf{B}_i^*(\varphi \vee \psi)$ is equivalent to a proposition of \mathbb{L}_0 , it follows that $\mathbf{B}_i^*(\varphi \vee \psi) \in \mathbb{K}$ iff $\mathbf{B}_i^*(\varphi \vee \psi) \in \mathbb{K}'$. Therefore $\mathbb{K} \cap \mathbb{L}_1 = \mathbb{K}' \cap \mathbb{L}_1$.

Suppose that $\mathbb{K} \cap \mathbb{L}_\alpha = \mathbb{K}' \cap \mathbb{L}_\alpha$, for $\alpha \geq 1$. Consider a proposition φ of depth $\alpha + 1$: it contains a set $\{\psi_1, \psi_2, \dots\}$ of well-formed formulas of depth 1 that begins with a modal operator \mathbf{B}_i^* . By **A5**, each ψ_i is equivalent to a proposition ψ'_i of \mathbb{L}_0 . If in the proposition φ , we replace all the ψ_i by the ψ'_i , we obtain a proposition φ' of depth α . By **A5** and **A2**, $\varphi \in \mathbb{K}$ iff $\varphi' \in \mathbb{K}$ as well as $\varphi \in \mathbb{K}'$ iff $\varphi' \in \mathbb{K}'$. Since $\varphi' \in \mathbb{K}$ iff $\varphi' \in \mathbb{K}'$, $\varphi \in \mathbb{K}$ iff $\varphi \in \mathbb{K}'$ and thus $\mathbb{K} \cap \mathbb{L}_{\alpha+1} = \mathbb{K}' \cap \mathbb{L}_{\alpha+1}$. ■

10 Appendix B

Theorem 2 : Consider two SESs H and $\overline{\mathcal{H}}$ and let the SYSs K and $\overline{\mathcal{K}}$ be their syntactical counterpart.

(1) If $H^* = H * \overline{\mathcal{H}}$ is a SES, then there exists a MSYS $K = (\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^*)$ which satisfies No Conflict **A4** and Belief and Message Inference **A5** and such that K^* is the syntactical

counterpart of H^* .

(2) If there exists a MSYS $K = (\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^*)$ which satisfies No Conflict **A4** and Belief and Message Inference **A5**, then $H^* = H * \mathcal{H}$ is a SES and is a semantical counterpart of K^* .

Proof. : (1) Let define the semantic structure \mathbb{H} as follows:

$$\mathbb{H} = \left(W, \overline{W}, W^*, (H_i)_{i \in I}, (\overline{H}_i)_{i \in I}, (H_i^*)_{i \in I}, S, \overline{S}, S^*, H_0, \overline{H}_0, H_0^*, \mathbf{w}^* \right)$$

with

- (i) H_i is an extended accessibility relation from $W \cup W^*$ toward $2^W \setminus \emptyset$, with $H_i(w, \overline{w}) = H_i(w)$,
- (ii) \overline{H}_i is an extended accessibility relation from $\overline{W} \cup W^*$ toward $2^{\overline{W}} \setminus \emptyset$, with $\overline{H}_i(w, \overline{w}) = \overline{H}_i(\overline{w})$,
- (iii) \overline{H}_0 is an extended accessibility relation of nature from $\overline{W} \cup W^*$ toward \overline{S} , with $\overline{H}_0(w, \overline{w}) = \overline{H}_0(\overline{w})$.

The truth value of a formula φ in some particular world (w, \overline{w}) is written: $(\mathbb{H}, (w, \overline{w})) \models \varphi$, which means that φ is true in (w, \overline{w}) within the structure \mathbb{H} . The following valuation rules express the truth assignment in W^* for any formula:

$$\left\{ \begin{array}{l} (\mathbb{H}, (w, \overline{w})) \models p \text{ iff } (\mathcal{H}^*, (w, \overline{w})) \models p, \\ (\mathbb{H}, (w, \overline{w})) \models \neg \varphi \text{ iff } (\mathbb{H}, (w, \overline{w})) \not\models \varphi, \\ (\mathbb{H}, (w, \overline{w})) \models \varphi \wedge \psi \text{ iff } (\mathbb{H}, (w, \overline{w})) \models \varphi \text{ and } (\mathbb{H}, (w, \overline{w})) \models \psi, \\ \left\{ \begin{array}{l} \text{for } \varphi \in \mathbb{L}, (\mathbb{H}, (w, \overline{w})) \models \mathbf{B}_i^* \varphi \\ \text{iff } (\mathbb{H}, (w', \overline{w}')) \models \varphi \text{ for all } (w', \overline{w}') \text{ s.t. } (w', \overline{w}') \in H_i^*(w, \overline{w}), \\ \text{for } \varphi \in \mathcal{L}, (\mathbb{H}, (w, \overline{w})) \models \varphi \text{ iff } (\mathcal{H}, w) \models \varphi, \\ \text{for } \varphi \in \overline{\mathcal{L}}, (\mathbb{H}, (w, \overline{w})) \models \varphi \text{ iff } (\overline{\mathcal{H}}, \overline{w}) \models \varphi. \end{array} \right. \end{array} \right.$$

We denote $|\varphi|^*$ the set $\{(w, \overline{w}) \in W^* \text{ s.t. } (\mathbb{H}, (w, \overline{w})) \models \varphi\}$, $|\varphi|$ the set $\{w \in W \text{ s.t. } (\mathcal{H}, w) \models \varphi\}$ and $|\overline{\varphi}|$ the set $\{\overline{w} \in \overline{W} \text{ s.t. } (\overline{\mathcal{H}}, \overline{w}) \models \varphi\}$. Let \mathbb{K} be the syntactical counterpart of \mathbb{H} : $\mathbb{K} = \{\varphi \in \mathbb{L} \text{ s.t. } (\mathbb{H}, (\mathbf{w}, \overline{\mathbf{w}})) \models \varphi\}$.

Step 1: We show that, if \mathcal{H}^* is a SES, then \mathbb{K} is a MSYS. Remark that $\mathbb{T} = \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{T}^*$. Hence, $\mathbb{T} \subset \mathbb{K}$. It is obvious from the validation rules that \mathbb{K} satisfies **C1**, **C2** and **R1**.

Step 2: We show now that, if \mathcal{H}^* is a SES, \mathbb{K} satisfies **A4**: $\bigwedge_{m \in \overline{\mathcal{L}}_0} (\overline{\mathbf{B}}_i m \rightarrow \neg \mathbf{B}_i \neg m)$.

Consider any $m \in \overline{\mathcal{L}}_0$. There exists an index subset $\mathbf{L}' \subset \mathbf{L}$ such that m is equivalent to $\bigvee_{l \in \mathbf{L}'} m_l$. Consider any $(w, \overline{w}) \in W^*$ and any player $i \in I$. We have that $H_i^*(w, \overline{w}) \neq \emptyset$. Suppose first that $(\mathbb{H}, (w, \overline{w})) \models \overline{\mathbf{B}}_i m$, that is $\overline{H}_i(\overline{w}) \subset \bigcup_{l \in \mathbf{L}'} \overline{M}_l$. Then $H_i^*(w, \overline{w}) \neq \emptyset$ implies that $H_i(w) \cap \bigcup_{l \in \mathbf{L}'} M_l \neq \emptyset$ and hence, $(\mathbb{H}, (w, \overline{w})) \models \neg \mathbf{B}_i \neg m$. Therefore, $(\mathbb{H}, (w, \overline{w})) \models \overline{\mathbf{B}}_i m \rightarrow \neg \mathbf{B}_i \neg m$. Now, if $(\mathbb{H}, (w, \overline{w})) \not\models \overline{\mathbf{B}}_i m$, then necessarily $(\mathbb{H}, (w, \overline{w})) \models \overline{\mathbf{B}}_i m \rightarrow \neg \mathbf{B}_i \neg m$. Since it is true for all $m \in \overline{\mathcal{L}}_0$, the validation rules imply that $(\mathbb{H}, (w, \overline{w})) \models \bigwedge_{m \in \overline{\mathcal{L}}_0} (\overline{\mathbf{B}}_i m \rightarrow \neg \mathbf{B}_i \neg m)$. Since **A4** holds in any world, it is common belief in the semantical counterpart \mathbb{K} of \mathbb{H} .

Step 3: We show now that if \mathcal{H}^* is a SES, \mathbb{K} satisfies **A5**: $\forall \varphi \in \mathcal{L}, \forall \psi \in \overline{\mathcal{L}}$,

$$\bigwedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \overline{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i (m_l \rightarrow \varphi) \vee \overline{\mathbf{B}}_i (m_l \rightarrow \psi)) \longleftrightarrow \mathbf{B}_i^* (\varphi \vee \psi).$$

(\rightarrow) Consider any $(w, \overline{w}) \in W^*$. Suppose first that

$$(\mathbb{H}, (w, \overline{w})) \models \bigwedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \overline{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i (m_l \rightarrow \varphi) \vee \overline{\mathbf{B}}_i (m_l \rightarrow \psi)).$$

The validation rules imply that for all $m_l \in \mathcal{M}$,

$$(\mathbb{H}, (w, \overline{w})) \models \mathbf{B}_i \neg m_l \vee \overline{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i (m_l \rightarrow \varphi) \vee \overline{\mathbf{B}}_i (m_l \rightarrow \psi)$$

and therefore either:

- $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i \neg m_l$, which implies that, for all $w' \in H_i(w)$, $(\mathcal{H}, w') \models \neg m_l$: since by definition,

$$H_i^*(w, \bar{w}) = \bigcup_{l'=0,1,\dots,L} ((H_i(w) \cap M_{l'}) \times (\bar{H}_i(\bar{w}) \cap \bar{M}_{l'})),$$

for all $(w', \bar{w}') \in H_i^*(w, \bar{w})$, $(\mathbb{H}, (w', \bar{w}')) \models \neg m_l$, and thus $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^* \neg m_l$, which also implies that $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow (\varphi \vee \psi))$,

- or $(\mathbb{H}, (w, \bar{w})) \models \bar{\mathbf{B}}_i \neg m_l$, which implies that, for all $\bar{w}' \in \bar{H}_i(w)$, $(\bar{\mathcal{H}}, \bar{w}') \models \neg m_l$, and thus similarly to the preceeding case, $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow (\varphi \vee \psi))$,
- or $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i(m_l \longrightarrow \varphi)$, which implies that, for all $w' \in H_i(w)$, $(\mathcal{H}, w') \models (m_l \longrightarrow \varphi)$. Remark that the semantic rule implies that $|m_l \longrightarrow \varphi|^* = (|m_l \longrightarrow \varphi| \times \bar{W}) \cap W^*$ and therefore, for all $(w', \bar{w}') \in H_i^*(w, \bar{w})$, $H_i^*(w, \bar{w}) \subset |m_l \longrightarrow \varphi|^*$. Thus, $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow \varphi)$ which in turn implies that $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow (\varphi \vee \psi))$,
- or $(\mathbb{H}, (w, \bar{w})) \models \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)$, which implies that, for all $\bar{w}' \in \bar{H}_i(w)$, $(\bar{\mathcal{H}}, \bar{w}') \models (m_l \longrightarrow \psi)$. Thus, similarly to the preceeding case, $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow (\varphi \vee \psi))$.

Thus, for all $m_l \in \mathcal{M}$, $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(m_l \longrightarrow (\varphi \vee \psi))$, and therefore $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(\varphi \vee \psi)$, which implies that

$$(\mathbb{H}, (w, \bar{w})) \models \wedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i(m_l \longrightarrow \varphi) \vee \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)) \longrightarrow \mathbf{B}_i^*(\varphi \vee \psi).$$

(\longleftarrow) Suppose now that $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(\varphi \vee \psi)$. Therefore, for all $(w', \bar{w}') \in H_i^*(w, \bar{w})$, $(\mathbb{H}, (w', \bar{w}')) \models \varphi \vee \psi$. Consider $m_l \in \mathcal{M}$ and let suppose that $(\mathbb{H}, (w, \bar{w})) \not\models (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l)$. Then, the two sets $A = H_i(w) \cap M_l$ and $B = H_i(w) \cap \bar{M}_l$ are both non-empty. The semantic rule implies that $A \times B = H_i^*(w, \bar{w}) \cap (M_l \times \bar{M}_l)$. $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(\varphi \vee \psi)$ implies that $A \times B \subset |\varphi \vee \psi|^*$. The semantic rule implies that

$$|\varphi \vee \psi|^* = |\varphi|^* \cup |\psi|^* = [(|\varphi| \times \bar{W}) \cap W^*] \cup [(W \times |\psi|) \cap W^*].$$

So, $A \times B \subset (|\varphi| \times \bar{W}) \cup (W \times |\psi|)$ implies that $A \subset |\varphi|$, and thus, for all $w' \in A$, $(\mathcal{H}, w') \models \varphi$ or $B \subset |\psi|$, and thus, for all $\bar{w}' \in B$, $(\bar{\mathcal{H}}, \bar{w}') \models \psi$. Hence, we have one of the two conditions:

- for all $(w', \bar{w}') \in H_i^*(w, \bar{w})$, either $(\mathcal{H}, w') \models \neg m_l$ or $(\mathcal{H}, w') \models \varphi$, which implies that $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i(m_l \longrightarrow \varphi)$,
- for all $(w', \bar{w}') \in H_i^*(w, \bar{w})$, either $(\bar{\mathcal{H}}, \bar{w}') \models \neg m_l$ or $(\bar{\mathcal{H}}, \bar{w}') \models \psi$, which implies that $(\mathbb{H}, (w, \bar{w})) \models \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)$.

Thus, we prove that if $(\mathbb{H}, (w, \bar{w})) \not\models (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l)$ then $(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i(m_l \longrightarrow \varphi) \vee \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)$. Hence,

$$(\mathbb{H}, (w, \bar{w})) \models \mathbf{B}_i^*(\varphi \vee \psi) \rightarrow (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i(m_l \longrightarrow \varphi) \vee \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)).$$

On the whole, we have proved that

$$(\mathbb{H}, (w, \bar{w})) \models \wedge_{m_l \in \mathcal{M}} (\mathbf{B}_i \neg m_l \vee \bar{\mathbf{B}}_i \neg m_l \vee \mathbf{B}_i(m_l \longrightarrow \varphi) \vee \bar{\mathbf{B}}_i(m_l \longrightarrow \psi)) \longleftrightarrow \mathbf{B}_i^*(\varphi \vee \psi).$$

Since **A5** holds in any world, this axiom is common belief in the semantical counterpart \mathbb{K} of \mathbb{H} .

Step 4: Note finally that \mathcal{K}^* is the syntactical counterpart of \mathcal{H}^* . Actually, for all $\varphi \in \mathcal{L}^*$, $(\mathbb{H}, (w, \bar{w})) \models \varphi$ iff $(\mathcal{H}^*, (w, \bar{w})) \models \varphi$. Since $\mathcal{K}^* = \mathbb{K} \cap \mathcal{L}^*$, then \mathcal{K}^* is the syntactical counterpart of \mathcal{H}^* .

(2) Suppose that $\mathbb{K} = (\mathcal{K}, \bar{\mathcal{K}}, \mathcal{K}^*)$ is a MSYS which satisfies **A4** on one side, **A5** on the other. Suppose (*ad absurdum*) that \mathcal{H}^* is not a SES. Then, there exists at least a world $(w, \bar{w}) \in W^*$ and a player $i \in I$ such that $H_i^*(w, \bar{w}) = \emptyset$. Then, it follows that, for any $l \in \mathbf{L} = \{0, \dots, L\}$, if $(\bar{H}_i(\bar{w}) \cap \bar{M}_l) \neq \emptyset$, then $(H_i(w) \cap M_l) = \emptyset$. Since $\{\bar{M}_l\}_{l=0, \dots, L}$ and $\{M_l\}_{l=0, \dots, L}$ are respectively partitions of \bar{W} and W , this means that there exist two nonempty index subsets $\mathbf{L}', \bar{\mathbf{L}}' \subset \mathbf{L}$, such that $\mathbf{L}' \cap \bar{\mathbf{L}}' = \emptyset$, $H_i(w) \subset \bigcup_{l \in \mathbf{L}'} M_l$ and $\bar{H}_i(\bar{w}) \subset \bigcup_{l \in \bar{\mathbf{L}}'} \bar{M}_l$. Let us note $m = \bigvee_{l \in \mathbf{L}'} m_l$. By transcription, $(\mathcal{H}, w) \models \mathbf{B}_i \neg m$ and $(\bar{\mathcal{H}}, \bar{w}) \models \bar{\mathbf{B}}_i m$. By connectedness of \mathcal{H}^* , there exists a finite sequence of players i_1, \dots, i_n and a finite sequence of worlds $(w_1, \bar{w}_1), \dots, (w_n, \bar{w}_n)$ such that $i_n = i$, $(w_1, \bar{w}_1) = (\mathbf{w}, \bar{\mathbf{w}})$, $(w_n, \bar{w}_n) = (w, \bar{w})$ and, for all $n-1 \geq k \geq 1$, $(w_{k+1}, \bar{w}_{k+1}) \in H_{i_k}^*(w_k, \bar{w}_k)$. Note that, for all $n-1 \geq k \geq 1$, $w_{k+1} \in H_{i_k}(w_k)$ and $\bar{w}_{k+1} \in \bar{H}_{i_k}(\bar{w}_k)$.

We check rigorously the proof for the two simplest cases, that is when $n = 1$ and $n = 2$, and more informally for $n > 2$.

When $n = 1$, we have $(\mathcal{H}, \mathbf{w}) \models \mathbf{B}_i \neg m$ and $(\bar{\mathcal{H}}, \bar{\mathbf{w}}) \models \bar{\mathbf{B}}_i m$ which means that $\mathbf{B}_i \neg m \in \mathcal{K}$ and $\bar{\mathbf{B}}_i m \in \bar{\mathcal{K}}$, and thus $\mathbf{B}_i \neg m \wedge \bar{\mathbf{B}}_i m \in \mathbb{K}$ which is a direct violation of **A4**.

When $n = 2$, there exists m_l such that $w \in M_l$, $\bar{w} \in \bar{M}_l$, and thus $(\mathcal{H}, w) \models m_l \wedge \mathbf{B}_i \neg m_l$ and $(\bar{\mathcal{H}}, \bar{w}) \models m_l \wedge \bar{\mathbf{B}}_i m_l$.

Therefore, $(\mathcal{H}, \mathbf{w}) \models \neg \mathbf{B}_{i_1} \neg m_l$, $(\mathcal{H}, \mathbf{w}) \models \neg \mathbf{B}_{i_1} \neg (m_l \wedge \mathbf{B}_i \neg m_l)$, $(\bar{\mathcal{H}}, \bar{\mathbf{w}}) \models \neg \bar{\mathbf{B}}_{i_1} \neg m_l$ and $(\bar{\mathcal{H}}, \bar{\mathbf{w}}) \models \neg \bar{\mathbf{B}}_{i_1} \neg (m_l \wedge \bar{\mathbf{B}}_i m_l)$. Thus

$$\neg \mathbf{B}_{i_1} \neg m_l \wedge \neg \bar{\mathbf{B}}_{i_1} \neg m_l \wedge \neg \mathbf{B}_{i_1} \neg (m_l \wedge \mathbf{B}_i \neg m_l) \wedge \neg \bar{\mathbf{B}}_{i_1} \neg (m_l \wedge \bar{\mathbf{B}}_i m_l) \in \mathbb{K}$$

which is equivalent to

$$\neg (\mathbf{B}_{i_1} \neg m_l \vee \bar{\mathbf{B}}_{i_1} \neg m_l \vee \mathbf{B}_{i_1} (m_l \longrightarrow \neg \mathbf{B}_i \neg m_l) \vee \bar{\mathbf{B}}_{i_1} (m_l \longrightarrow \neg \bar{\mathbf{B}}_i m_l)) \in \mathbb{K}$$

i.e.

$$\neg \left(\frac{\mathbf{B}_{i_1} \neg m_l \vee \bar{\mathbf{B}}_{i_1} \neg m_l \vee \mathbf{B}_{i_1} (m_l \longrightarrow (\neg m_l \vee \neg \mathbf{B}_i \neg m_l)) \vee}{\bar{\mathbf{B}}_{i_1} (m_l \longrightarrow (\neg m_l \vee \neg \bar{\mathbf{B}}_i m_l))} \right) \in \mathbb{K}.$$

Since, for all $l' \neq l$, $\mathbf{B}_{i_1} (m_{l'} \longrightarrow (\neg m_l \vee \neg \mathbf{B}_i \neg m_l)) \in \mathbb{K}$ and $\bar{\mathbf{B}}_{i_1} (m_{l'} \longrightarrow (\neg m_l \vee \neg \bar{\mathbf{B}}_i m_l)) \in \mathbb{K}$, then

$$\bigvee_{m_{l'}} \neg \left(\frac{\mathbf{B}_{i_1} \neg m_{l'} \vee \bar{\mathbf{B}}_{i_1} \neg m_{l'} \vee \mathbf{B}_{i_1} (m_{l'} \longrightarrow (\neg m_l \vee \neg \mathbf{B}_i \neg m_l)) \vee}{\bar{\mathbf{B}}_{i_1} (m_{l'} \longrightarrow (\neg m_l \vee \neg \bar{\mathbf{B}}_i m_l))} \right) \in \mathbb{K}$$

and **A5** implies that $\neg \mathbf{B}_{i_1}^* ((\neg m_l \vee \neg \mathbf{B}_i \neg m_l) \vee (\neg m_l \vee \neg \bar{\mathbf{B}}_i m_l)) \in \mathbb{K}$ which is equivalent to $\neg \mathbf{B}_{i_1}^* \neg (m_l \wedge \mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l) \in \mathbb{K}$. Otherwise, by **A4**, $\mathbf{B}_{i_1}^* \neg (\mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l) \in \mathbb{K}$. Besides, since $\neg (\mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l) \rightarrow \neg (m_l \wedge \mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l)$ is a tautology, then

$$\mathbf{B}_{i_1}^* (\neg (\mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l) \rightarrow \neg (m_l \wedge \mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l)) \in \mathbb{K}.$$

Finally, by **A2**, we get: $\mathbf{B}_{i_1}^* \neg (m_l \wedge \mathbf{B}_i \neg m_l \wedge \bar{\mathbf{B}}_i m_l) \in \mathbb{K}$ which is a contradiction.

For $n > 2$, there exists a sequence $l_1, \dots, l_{n-1} \in \mathbf{L}$ such that

$$(\mathcal{H}, \mathbf{w}) \models \neg \mathbf{B}_{i_1} \neg (m_{l_1} \wedge \neg \mathbf{B}_{i_2} \neg (\dots \wedge \neg \mathbf{B}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m)))$$

and

$$(\overline{\mathcal{H}}, \overline{\mathbf{w}}) \models \neg \overline{\mathbf{B}}_{i_1} \neg (m_{l_1} \wedge \neg \overline{\mathbf{B}}_{i_2} \neg (\dots \wedge \neg \overline{\mathbf{B}}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \overline{\mathbf{B}}_i m))).$$

Similarly, according to **A5**, we can prove first that

$$\neg \mathbf{B}_{i_1}^* \neg \left(\begin{array}{c} m_{l_1} \wedge \neg \mathbf{B}_{i_2} \neg (\dots \wedge \neg \mathbf{B}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m)) \wedge \\ \neg \overline{\mathbf{B}}_{i_2} \neg (\dots \wedge \neg \overline{\mathbf{B}}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \overline{\mathbf{B}}_i m)) \end{array} \right) \in \mathbb{K}$$

then that

$$\neg \mathbf{B}_{i_1}^* \neg \left(\begin{array}{c} m_{l_1} \wedge \neg \mathbf{B}_{i_2}^* \neg (m_{l_2} \wedge \neg \mathbf{B}_{i_3} \neg (\dots \wedge \neg \mathbf{B}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m)) \wedge \\ \neg \overline{\mathbf{B}}_{i_3} \neg (\dots \wedge \neg \overline{\mathbf{B}}_{i_{n-1}} \neg (m_{l_{n-1}} \wedge \overline{\mathbf{B}}_i m)) \end{array} \right) \in \mathbb{K}$$

until that $\neg \mathbf{B}_{i_1}^* \neg (m_{l_1} \wedge \neg \mathbf{B}_{i_2}^* \neg (\dots \wedge \neg \mathbf{B}_{i_{n-1}}^* \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m))) \in \mathbb{K}$ which can be rewritten as:

$$\neg \mathbf{B}_{i_1}^* \left(\neg m_{l_1} \vee \mathbf{B}_{i_2}^* (\dots \vee \mathbf{B}_{i_{n-1}}^* \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m)) \right) \in \mathbb{K}.$$

Otherwise, by **A4**, $\mathbf{B}_{i_1}^* \dots \mathbf{B}_{i_{n-1}}^* \neg (\mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m) \in \mathbb{K}$. By using the rule of classical logic and **A2**, we can successively prove that

$$\mathbf{B}_{i_1}^* \dots \mathbf{B}_{i_{n-1}}^* \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m) \in \mathbb{K},$$

$$\mathbf{B}_{i_1}^* \dots \mathbf{B}_{i_{n-2}}^* \left(\neg m_{l_{n-2}} \vee \mathbf{B}_{i_{n-1}}^* \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m) \right) \in \mathbb{K},$$

until $\mathbf{B}_{i_1}^* \left(\neg m_{l_1} \vee \mathbf{B}_{i_2}^* (\dots \vee \mathbf{B}_{i_{n-2}}^* \left(\neg m_{l_{n-2}} \vee \mathbf{B}_{i_{n-1}}^* \neg (m_{l_{n-1}} \wedge \mathbf{B}_i \neg m \wedge \overline{\mathbf{B}}_i m) \right) \right) \in \mathbb{K}$ which yields a contradiction. Since \mathcal{H}^* is a SES, as proved just above, we can construct a semantic structure \mathbb{H} as in the proof of (1). The syntactical counterpart of \mathbb{H} is a MSYS $\mathbb{K}' = (\mathcal{K}', \overline{\mathcal{K}}', \mathcal{K}'^*)$ such that

- $\mathcal{K}' = \mathcal{K}$ since \mathcal{K}' and \mathcal{K} are the syntactical counterpart of \mathcal{H} ,
- $\overline{\mathcal{K}}' = \overline{\mathcal{K}}$ since $\overline{\mathcal{K}}'$ and $\overline{\mathcal{K}}$ are the syntactical counterpart of $\overline{\mathcal{H}}$,
- \mathcal{K}'^* is the syntactical counterpart of \mathcal{H}^* ,
- \mathbb{K}' satisfies **A4** and **A5**.

By Theorem 1, $\mathbb{K}' = \mathbb{K}$, and thus \mathcal{K}^* is the syntactical counterpart of \mathcal{H}^* . ■